Pratt

Math 150 – Fall 2020
Algebra & Trigonometry

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Session 13: Monday 11/30/20
6:30pm - 9:20pm
via REMOTE LEARNING
Revision 1

Instructor Contact Information

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Fall 2020 Virtual Office hours *ONLY*
Wednesdays 10:00am-2:00pm via Zoom Meeting
To make your appointment
Send me an email at least one day in advance :
crubenst@pratt.edu
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Math 150 – Class Topics

1. **The Foundations of Algebra**
2. **Equations and Inequalities**
3. **Functions**
4. **Polynomial Functions**
5. **Rational Functions and Conic Sections**
6. **Exponential and Logarithmic Functions**
7. **The Trigonometric Functions**
8. **Analytic Trigonometry**
9. **Applications of Trigonometry**
10. **Systems of Equations and Inequalities**
11. **Matrices, Linear Systems, and Determinants**
12. **Topics in Algebra**

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Draft Schedule: Math 150 – Fall 2020 – Remote Learning

Monday	Notes
24-Aug	1. Introduction: Numbers, Arithmetic Operations, Fractions
31-Aug	2. Manipulation of Algebraic Expressions; *Hwk #1 Due @ Noon*
7-Sep	NO CLASSES – Labor Day
14-Sep	3. Solving Linear and Quadratic Equations of One Variable; *Hwk #2 Due*
21-Sep	4. Solving Equations of Two Variables; *Hwk #3 Due*
28-Sep	NO CLASSES – Instructor Holiday
5-Oct	5. Creating Equations: Polynomials, Exponents & Logarithms *Hwk #4 Due*
12-Oct	6. Functions, Graphing, Exponents and Logarithms; *Hwk #5; Exam #1*
19-Oct	7. Logarithmic and Exponential Models; *Hwk #6; Exam #1 Review*
26-Oct	8. Trigonometric Functions, Pythagorean Theorem; *Hwk #7 Due*
2-Nov	9. Applications of Trigonometry; *Hwk #8 Due*
9-Nov	10. Analytic Trigonometry: Identities & Graphing; *Hwk #9 Due; Exam #2*
16-Nov	11. Areas and Volumes of Geometric Solids; *Hwk #10; Exam #2 Review*
23-Nov	12. Systems of Equations and Inequalities
30-Nov	13. Series and Sequences, Review topics
7-Dec	Final Examination *Emailed Monday by 6:00pm ET - Due at 10:00pm ET*

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www.CharlesRubenstein.com/150

20fa13.pdf = *This slide set**
20fa13_h.pdf = *slides as 6-up handouts**

**My goal is to post these not later than Noon on the Friday or Saturday before our Zoom Class Meetings*

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Questions?

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This Class Session #13

***In class* – Session 13: Monday 30 November:**

- *Review:* ***Homework Set #12 & #13; Textbook readings***
- ***Lecture: Series and Sequences; Review Topics***

Next Class:

Monday 7 December

In-class "2-hour" FINAL EXAM

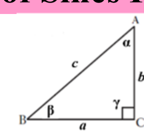
Emailed to the class not later than 6:00pm

Must be returned to me not later than 10:00pm

Questions?

About the Law of Sines and Law of Cosines and Angle Measurements…

Law of Sines Review

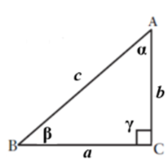


$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Or, alternatively…

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines Review



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ab\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Angle Measurement Review

We can 'easily' express angle as whole numbers, but when it comes to fractional degrees how do we handle those?

We also know that we can express fractions of a degree using the "minutes" and "seconds" method.

But what if your calculator only handles fractions of a degree and the problem has minutes and seconds?

1 degree = 60 minutes *and* ***1 minute = 60 seconds***

Thus

$$1\ \text{minute} = \frac{1\ \text{degree}}{60} = 0.0167\ \text{degrees}$$

$$1\ \text{second} = \frac{1\ \text{minute}}{60} = \frac{0.0167\ \text{degrees}}{60} = 0.00278\ \text{degrees}$$

About Radians and Degrees:

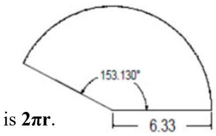
2π radians = 360° *therefore* **1 radian = 57.2958°** *and*

1° = 0.01745 radians

Homework #12
Areas and Volumes

Homework #12 Review

1. Find the length of the arc:



We know that the circumference of a circle is $\mathbf{2\pi r}$.

$\mathbf{2\pi = 360^o}$ therefore **1 radian = 57.2958°**

Thus the length of the arc (perimeter or circumference) would be:

$$L = \frac{\emptyset}{57.2958} r$$
$$= \frac{153.13}{57.2958}(6.33)$$
$$L = 2.6726\,(6.33)$$
$$L = 16.9177$$

Homework #12 Review

2. Find the **volume** of a **cone** with a circular base of radius = **2** and height = **9**.

Volume of a Cone, $V = \frac{1}{3}(\pi r^2 h)$

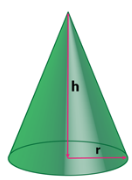
$V = \frac{1}{3}[\pi(2)^2(9)]$

$V = \frac{1}{3}[3.1415(4)(9)]$

$V = \frac{1}{3}[3.1415(36)]$

$V = \frac{1}{3}[113.094]$

$V = 37.698$



Homework #12 Review

3. Find the **volume** of a **pyramid** whose base is an equilateral triangle of side = **3** and height = **8**

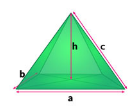
Volume of a pyramid, $V = \frac{1}{3} a^2 h$

$V = [\frac{1}{3}(3)^2](8)$

$= [\frac{1}{3}(9)](8)$

$= [3](8)$

$V = 24$



Homework #12 Review

4. Find the volume of a pyramid with a circular base of radius **1** and height = **8** if the base is in the *x-y* plane, centered on **0,0,0** and the tip is at $(x, y, z) = (\mathbf{4, 3, 10})$.

A cone is a pyramid with a circular base with radius **r** and height **h**.

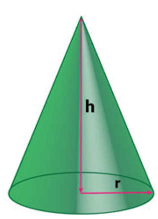
Volume of a Cone $= \frac{1}{3}(\pi r^2 h)$

$V = \frac{1}{3}(\pi r^2 h)$

$= \frac{1}{3}[\pi(1)^2(8)]$

$= \frac{1}{3}[\pi(8)] = \frac{8}{3}\pi = 8.3773$

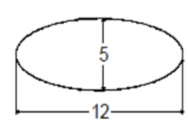
$V = 8.3773$



Homework #12 Review

5. Calculate the area of an ellipse with height = **5** and width = **12**.

Area of an ellipse, $A = \pi r_1 r_2$



$A = \pi r_1 r_2$

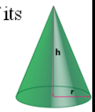
$= \pi(5)(12)$

$= 60\,\pi$

$A = 188.49$

Homework #12 Review

6. When the radial lines on the figure below are brought together, a cone is formed. Find the height of the cone and the radius of its circular base.



The arc angle $\emptyset$ = **360º – 32º** thus $\emptyset$ **= 328º**

1 radian = 57.2958º → 328º / 57.2958º and $\emptyset$ **= 5.7247 rad**

The length of the arc equals the circumference of the base of the cone:

$\mathbf{L} = \frac{\emptyset}{57.2958}\,\mathbf{r} = \frac{328}{57.2958}(5) = 5.7247\ (5)$ thus $\mathbf{L = 28.6234}$

$\mathbf{L} = 28.6234 = 2\pi\mathbf{r}$ thus $\mathbf{r} = \frac{28.6234}{2\pi}$ or $\mathbf{r = 4.5555}$

We find the height (**h**) using Pythagorean's Theorem with **s=5**:

$h^2 + r^2 = s^2 \rightarrow h^2 = s^2 - r^2 = (5)^2 - (4.5555)^2 = 25 - 20.753$

$h^2 = 4.247$ thus, $h = 2.06$

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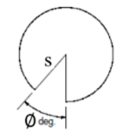
Homework #12 Review

7. Specify the figure, similar to the figure of Problem 6 which will form a cone of radius (**r**) = **5** and height (**h**) =**10**.

Find the angle and the sector radius.



Use Pythagorean's Theorem find the sector radius (**s**):

$h^2 + r^2 = s^2 \rightarrow (10)^2 + (5)^2 = s^2 = (125);\ s = 11.18$

The length of the arc equals the circumference of the base of the cone (*2πr = 10π = 31.4159 and 1 radian = 57.2958º*):

$\mathbf{L} = \frac{\emptyset}{57.2958}\,\mathbf{s} = 2\pi\mathbf{r} \rightarrow \frac{\emptyset}{57.2958}(11.18) = 31.4159$

$11.18\ \emptyset = 31.4159\ (57.2958) \rightarrow 11.18\ \emptyset = 1800$

Thus, $\emptyset$ = **1800 / 11.18** and the arc angle $\emptyset$ **= 161º**

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Homework #12 Review

8. Calculate the **volume** of a **spherical shell** with an outer diameter of **3** inches and a thickness of **1/4** inch.

Volume of a sphere = $\frac{4}{3}\pi r^3$

Here we have two spheres: the outer has a diameter ($\mathbf{d_1}$) of 3.0" and the inner a diameter ($\mathbf{d_2}$) of (3-¼-¼) inches = 2.5"

The two volumes need to be subtracted from one another:

Shell Volume = $V_s = V_{outer} - V_{inner} = \frac{4}{3}\pi(r_1)^3 - \frac{4}{3}\pi(r_2)^3$

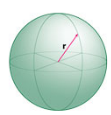
$V_s = \frac{4}{3}\pi(r_1)^3 - \frac{4}{3}\pi(r_2)^3 = \frac{4}{3}\pi[(r_1)^3 - (r_2)^3]$

NOTE: $\frac{1}{2} d_1 = r_1$ and $\frac{1}{2} d_2 = r_2$ *yielding* $r_1 = 1.5$" and $r_2 = 1.25$"

$V_s = \frac{4}{3}\pi[(1.5)^3 - (1.25)^3]$

$= \frac{4}{3}\pi[(3.375) - (1.953)]$

$= \frac{4}{3}\pi[1.422]$



$V_s = 5.956$ **square inches!**

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Homework #12 Review

9. Find the **radius** of the **sphere** (r_s) whose **volume** is equal to that of **cone** of height = **6** and radius $r_c = 4$.

Volume of a Sphere = $\frac{4}{3}\pi r_s^3$

Volume of a Cone = $\frac{1}{3}(\pi r_c^2 h)$

Therefore: $\frac{4}{3}\pi r_s^3 = \frac{1}{3}(\pi r_c^2 h)$ *and we need to solve for* r_s

$r_s^3 = [\frac{1}{3}(\pi r_c^2 h)] / \frac{4}{3}\pi$

multiply right side by **3/3**

$r_s^3 = (\pi r_c^2 h) / 4\pi$

now cancel the two **π***'s* :

$r_s^3 = \frac{1}{4}(r_c^2 h)$

Substituting the cone values $r_c^2 = 4$ **and** $h = 6$**:**

$r_s^3 = \frac{1}{4}[(4)^2(6)] = r_s^3 = 24$

Thus $r_s = 2.885$

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Homework #12 Review

10. Find (a formula for) the radius ***R*** of the **sphere** whose **volume** is equal the **volume** of a **cone** with radius ***r*** and height ***h***.

Very similar to #9…

Volume of a Sphere = $\frac{4}{3}\pi R^3$

Volume of a Cone = $\frac{1}{3}(\pi r^2 h)$

Therefore: $\frac{4}{3}\pi R^3 = \frac{1}{3}(\pi r_c^2 h)$ and we need to solve for **R**

$R^3 = [\frac{1}{3}(\pi r^2 h)] / \frac{4}{3}\pi$

multiply right side by **3/3**

$R^3 = (\pi r^2 h) / 4\pi$

now cancel the two **π***'s* :

$R^3 = \frac{1}{4}(r^2 h)$

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Questions?

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Arithmetic Series and Sequences

Sequences

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English.

Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Two examples are

1, 2, 3, 4, . . . and
1, 3, 5, 7, ...

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers.

Rather than using function notation, however, sequences are usually written using subscript notation.

Sequences

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers.

The function values

$$a_1,\ a_2,\ a_3,\ a_4, \dots, a_n, \dots.$$

are the **terms** of the sequence. When the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with **0** instead of **1** so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \ \dots$$

When this is the case, the domain includes **0**.

Terms of a Sequence

a. The first four terms of the sequence given by

$$a_n = 3n - 2$$

are

$a_1 = 3(1) - 2 = 1$ 1st term
$a_2 = 3(2) - 2 = 4$ 2nd term
$a_3 = 3(3) - 2 = 7$ 3rd term
$a_4 = 3(4) - 2 = 10$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$$a_1 = 3 + (-1)^1 = 3 - 1 = 2$$

$$a_2 = 3 + (-1)^2 = 3 + 1 = 4$$

$$a_3 = 3 + (-1)^3 = 3 - 1 = 2$$

$$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$$

Sequences

Simply listing the first few terms is not sufficient to define a unique sequence—the ***n*th** term *must be given*.

To see this, consider the following sequences, both of which have the ***same first three terms***.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^n}, \dots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \dots, \frac{6}{(n+1)(n^2 - n + 6)}, \dots$$

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms.

All other terms of the sequence are then defined using previous terms.

A well-known recursive sequence is the ***Fibonacci*** *sequence.*

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

- **Definition of Factorial**

If n is a positive integer, then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \ \cdot 3 \ \cdot 4 \ \cdots (n-1) \ \cdot n.$$

As a special case, ***zero factorial*** is defined as **0! = 1**

Notice that **0! = 1** and **1! = 1**

Here are some other values of ***n*!**

2! = 1 · 2 = 2

3! = 1 · 2 · 3 = 6

4! = 1 · 2 · 3 · 4 = 24

Factorials follow the same conventions for order of operations as do exponents. For instance, $2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$

$$(2n)! = 1 \cdot 2 \ \cdot 3 \ \cdot 4 \ \cdots 2n.$$

Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}.$$

Begin with ***n* = 0**.

Solution:

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \qquad a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \qquad a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \qquad a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \qquad a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$$

When working with fractions involving factorials, you will often be able to reduce the fractions to simplify the computations.

Summation Notation

A convenient notation for the sum of the terms of a finite sequence is called **summation notation** or **sigma notation**. It involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first *n* terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where ***i*** is called the **index of summation, *n*** is the **upper limit of summation,** and **1** is the **lower limit of summation.**

Summation Notation for a Sum

a. $\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) = 45$

b. $\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2) = 10 + 17 + 26 + 37 = 90$

c. $\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320} \approx 2.71828$

*For this summation, note that the sum is very close to the irrational number e ≈ 2.718281828. It can be shown that as more terms of the sequence whose **n**th term is* $\frac{1}{n!}$ *are added, the sum becomes closer and closer to **e**.*

Properties of Sums

1. $\sum_{i=1}^{n} c = c$, c is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Questions?

Arithmetic Series and Sequences

Series

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

- **Definition of Series**

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$

The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots \sum_{i=1}^{\infty} a_i.$$

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Finding the Sum of a Series

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$ find: (a) the third partial sum and (b) the sum.

- Solution:

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333$$

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots = 0.33333\ldots = \frac{1}{3}$$

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Compound Interest

An investor deposits **$5000** in an account that earns **3%** interest compounded quarterly.

The balance in the account after **n** quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 0,\ 1,\ 2, \ldots$$

a. Write the first three terms of the sequence.

$$A_0 = 5000\left(1 + \frac{0.03}{4}\right)^0 = \$5000.00$$

$$A_1 = 5000\left(1 + \frac{0.03}{4}\right)^1 = \$5037.50$$

$$A_2 = 5000\left(1 + \frac{0.03}{4}\right)^2 = \$5075.28$$

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Compound Interest

An investor deposits $5000 in an account that earns 3% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 0, 1, 2, \ldots$$

b. Find the balance in the account after **10 years** by computing the ***40th term of the sequence***.

$$A_{40} = 5000\left(1 + \frac{0.03}{4}\right)^{40} = \$6741.74$$

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Questions?

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Arithmetic Series and Sequences

Arithmetic Sequences

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Arithmetic Sequences

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same.

So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

where d is the **common difference** of the arithmetic sequence

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Arithmetic Sequences: Examples

a. The sequence whose **nth** term is $4n + 3$ is arithmetic.

For this sequence, the common difference between consecutive terms is 4.

Begin with $n = 1$: $\underbrace{7, 11}_{11-7=4}, 15, 19, \ldots, 4n+3, \ldots$

b. The sequence whose **nth** term is $7 - 5n$ is arithmetic. For this sequence, the common difference between consecutive terms is -5.

Begin with $n = 1$: $\underbrace{2, -3}_{-3-2=-5}, -8, -13, \ldots, 7-5n, \ldots$

c. The sequence whose **nth** term is $\frac{1}{4}(n+3)$ is arithmetic.

For this sequence, the common difference between consecutive terms is $\frac{1}{4}$.

Begin with $n = 1$: $\underbrace{1, \frac{5}{4}}_{\frac{5}{4}-1=\frac{1}{4}}, \frac{3}{2}, \frac{7}{4}, \ldots, \frac{n+3}{4}, \ldots$

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Non-Arithmetic Sequences

The sequence **1**, **4**, **9**, **16**, . . . , whose **nth** term is n^2 is *not* arithmetic. The difference between the first two terms is $a_2 - a_1 = 4 - 1 = 3$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5$$

The **nth** term of an arithmetic sequence can be derived from the following pattern.

$a_1 = a_1$ 1st term

$a_2 = a_1 + d$ 2nd term

$a_3 = a_1 + 2d$ 3rd term.

$a_4 = a_1 + 3d$ 4th term.

$a_5 = a_1 + 4d$ 5th term.

1 less

⋮

$a_n = a_1 + (n-1)d$ nth term.

1 less

n 45

Arithmetic Sequences

The nth Term of an Arithmetic Sequence

The **nth** term of an arithmetic sequence has the form

$$a_n = a_1 + (n-1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term.

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Arithmetic Sequence Example

Find a formula for the **nth** term of the arithmetic sequence whose common difference is **3** and whose first term is **2**.

- Solution:

The formula for the **nth** term is of the form

$$a_n = a_1 + (n-1)d$$

Substituting the common difference $d = 3$ and the first term $a_1 = 2$,

the formula must have the form $a_n = 2 + 3(n-1)$

and the formula for the nth term is

$$a_n = 3n - 1$$

The sequence therefore has the form:

$$2, 5, 8, 11, 14, \ldots, 3n-1, \ldots$$

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Arithmetic Sequence Example

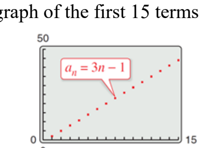
The formula for the **nth** term is thus

$$a_n = 3n - 1$$

The sequence therefore has the form:

$$2, 5, 8, 11, 14, \ldots, 3n-1, \ldots$$

The figure shows a graph of the first 15 terms of the sequence:



Notice that the points all lie on a line.

This makes sense because a_n *(y axis)* is a linear function of n *(x axis)*.

NOTE: the terms "***arithmetic***" and "***linear***" are closely connected.

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The Recursion Formula

When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the **$(n + 1)$th** term by using the *recursion formula:*

$$a_{n+1} = a_n + d$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term.

For instance, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Sum of a Finite Arithmetic Sequence

There is also a formula for the *sum* of a finite arithmetic sequence.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Find the sum: **1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19**

- Solution:

To begin, notice that the sequence is arithmetic (*with a common difference of 2*) and has 10 terms. So, the sum of the sequence is

$$s_n = \frac{n}{2}(a_1 + a_n)$$

Substituting 10 for n, 1 for a_1, and 19 for a_n yields $S_n = \frac{10}{2}(1 + 19)$

which Simplifies to $S_n = 100$

*The sum of the first **n** terms of an infinite sequence is the **nth partial sum**.*

Total Sales Example - 1

A small business sells **$10,000** worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by **$7,500** each year for **9 years**. Assuming that this goal is met, find the total sales during the **first 10 years** this business is in operation.

- Solution: The annual sales form an arithmetic sequence in which

$a_1 = 10{,}000$ and $d = 7500$.

Thus, $a_n = 10{,}000 + 7500(n - 1)$

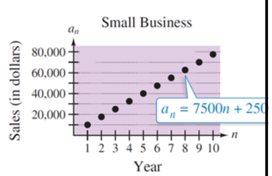
and the **nth** term of the sequence is

$a_n = 7500n + 2500$

Therefore, the ***10th term*** is

$a_{10} = 7500(10) + 2500 = 77{,}500$

See figure on right…



Total Sales Example - 2

… Assuming that this goal is met, ***find the total sales during the first 10 years*** this business is in operation.

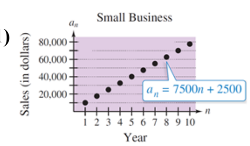
- Solution: $a_n = 10{,}000 + 7500(n - 1)$

and the **nth** term of the sequence is

$a_n = 7500n + 2500$

Therefore, the ***10th term*** is

$a_{10} = 7500(10) + 2500 = 77{,}500$



The sum of the first 10 terms of the sequence is $S_{10} = \frac{n}{2}(a_1 + a_{10})$

$$= \frac{10}{2}(10{,}000 + 77{,}500)$$

$$= 5(87{,}500) = 437{,}500$$

So, the total sales for the first 10 years will be **$437,500**.

Questions?

Arithmetic Series and Sequences

Geometic Series and Sequences

Geometric Sequences

Sequences whose consecutive terms have a common ***difference*** are called **arithmetic sequences**.

Sequences whose consecutive terms have a common ***ratio*** are called **geometric sequences**.

- **Definition of Geometric Sequence**

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ is geometric when there is a number r such that $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \quad r \neq 0.$

The number r is the **common ratio** of the geometric sequence.

A geometric sequence may be thought of as an ***exponential function*** *whose domain is the set of natural numbers.*

nth Term - Geometric Sequences

- **The *n*th Term of a Geometric Sequence**

The ***n*th** term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence.

Every geometric sequence can be written in the following form.

$$a_1, \; a_2, \; a_3, \; a_4, \; a_5, \; \dots, \; a_n, \dots$$
$$\downarrow$$
$$a_1, \; a_1r, \; a_1r^2, \; a_1r^3, \; a_1r^4, \; \dots, \; a_1r^{n-1}, \dots$$

When you know the ***n*th** term of a geometric sequence, you can find the $(n + 1)^{th}$ term by multiplying by r. That is,

$$a_{n+1} = a_n r$$

Geometric Sequences - Example

Find the 12th term of the geometric sequence

5, 15, 45, ...

- Solution:

The common ratio of this sequence is $r = \frac{15}{5} = 3.$

Because the first term is $a_1 = 5$, the **12th** term $(n = 12)$ is found:

Formula for *n*th term of a geometric sequence

$$a_n = a_1 r^{n-1}$$

Substitute 5 for a_1, 3 for r, and 12 for n:

$$a_{12} = 5(3)^{12-1}$$

$$a_{12} = 5\,(177{,}147) = 885{,}735$$

NOTE: When you know any two terms of a geometric sequence, you can use that information to find any other term of the sequence.

Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

- **The Sum of a Finite Geometric Sequence**

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \dots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

Example

Sum of a Finite Geometric Sequence

Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}.$

- Solution: $\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right]$

You have $\sum_{i=1}^{12} 4(0.3)^{i-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + \cdots + 4(0.3)^{11}.$

Now, $a_1 = 4$, $r = 0.3$, and $n = 12$, so applying the formula for the sum of a finite geometric sequence, you obtain

Sum of a finite geometric sequence $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

Substitute 4 for a_1, 0.3 for r, and 12 for n.

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right]$$

$$\approx 5.714$$

Sum of a Finite Geometric Sequence

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form where the exponent for r is $i - 1$. $\sum_{i=1}^{n} a_1 r^{i-1}$

For a sum that is not of this form, you must adjust the formula.

For instance, if the sum were $\sum_{i=1}^{12} 4(0.3)^i$

then you would evaluate the sum as follows:

$$\sum_{i=1}^{12} 4(0.3)^i = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12}$$

$$= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdots + [4(0.3)](0.3)^{11}$$

$$= 4(0.3)\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right]$$

which, for $a_1 = 4(0.3)$, $r = 0.3$, $n = 12$

$$\sum_{i=1}^{12} 4(0.3)^i \approx 1.714$$

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*.

Specifically, if the common ratio r has the property that $|r| < 1$,

it can be shown that r^n becomes arbitrarily close to zero as n increases without bound.

As a result:

$$a_1\left(\frac{1-r^n}{1-r}\right) \to a_1\left(\frac{1-0}{1-r}\right) \quad \text{as} \quad n \to \infty.$$

Geometric Series

If $a_1\left(\frac{1-r^n}{1-r}\right) \to a_1\left(\frac{1-0}{1-r}\right)$ as $n \to \infty$.

The following summarizes this result.

- **The Sum of an Infinite Geometric Series**

If $|\mathbf{r}| < 1$ then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum $S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1-r}$.

Note that if $|\mathbf{r}| \geq 1$ the series does ***not*** have a sum.

When $\mathbf{r} = 1$ the equation has the form **0/0** which is undefined…

Sum of an Infinite Geometric Series

Find each sum:

a. $\sum_{n=0}^{\infty} 4(0.6)^n$

Solution: $\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^n + \cdots$

$$\frac{a_1}{1-r} = \frac{4}{1-0.6} = 10$$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution:
$3 + 0.3 + 0.03 + 0.003 + \cdots = 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots$

$$\frac{a_1}{1-r} = \frac{3}{1-0.1} = \frac{10}{3} \approx 3.33$$

Increasing Annuity Example - 1

An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an **increasing annuity.**)

- Solution:

To find the balance in the account after 24 months, consider each of the 24 deposits separately.

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1+\frac{0.03}{12}\right)^{24} = 50(1.0025)^{24}$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1+\frac{0.03}{12}\right)^{23} = 50(1.0025)^{23}$$

Increasing Annuity Example - 1

… What is the balance at the end of 2 years?

- Solution: *continued*

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1+\frac{0.03}{12}\right)^{1} = 50(1.0025)$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.0025)$ and $r = 1.0025$, and $n = 24$, you have

$$S_n = A_1\left(\frac{1-r^n}{1-r}\right)$$

Substitute 50(1.005) for A_1, 1.005 for r, and 24 for n.

$$S_{24} = 50(1.0025)\left[\frac{1-(1.0025)^{24}}{1-1.0025}\right] \approx \$1238.23$$

Questions?

Next Class Session #14

Don't Forget to do the Online Course Evaluation!

<u>***In class*** **– Session 14: Monday 7 December:**</u>

"In-class"

FINAL EXAM

(Worth 20% of Final Grade)

Will be

Emailed to you not later than 6:00pm

and is

Due back to me not later than 10:00pm

On Monday 7 December 2020

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Any Questions?
Send me an email …

crubenst@pratt.edu
or
c.rubenstein@ieee.org

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End

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